

# Martingale representation in progressive enlargement by the reference filtration of a semi-martingale: a note on the multidimensional case

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## Abstract

Let  $\mathbf{X}$  and  $\mathbf{Y}$  be an  $m$ -dimensional  $\mathbb{F}$ -semi-martingale and an  $n$ -dimensional  $\mathbb{H}$ -semi-martingale respectively on the same probability space  $(\Omega, \mathcal{F}, P)$ , both enjoying the strong predictable representation property. We propose a martingale representation result for the square-integrable  $(P, \mathbb{G})$ -martingales, where  $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ . As a first application we identify the biggest possible value of the multiplicity in the sense of Davis and Varaiya of  $\bigvee_{i=1}^d \mathbb{F}^i$ , where, fixed  $i \in (1, \dots, d)$ ,  $\mathbb{F}^i$  is the reference filtration of a real martingale  $M^i$ , which enjoys the  $(P, \mathbb{F}^i)$  predictable representation property. A second application falls into the framework of credit risk modeling and in particular into the study of the progressive enlargement of the market filtration by a default time. More precisely, when the risky asset price is a multidimensional semi-martingale enjoying the strong predictable representation property and the default time satisfies the density hypothesis, we present a new proof of the analogous of the classical Kusuoka's theorem.

**Keywords:** Semi-martingales, predictable representations property, enlargement of filtration, completeness of a financial market

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# 1 Introduction

On a stochastic basis  $(\Omega', \mathcal{F}', \mathbb{F}', P')$  let  $\mathbf{X}' = (X'^1, \dots, X'^l)$  be an  $l$ -dimensional  $\mathbb{F}'$ -semi-martingale, which admits at least one equivalent local-martingale measure  $Q'$ . Then  $\mathbf{X}'$  enjoys the  $(Q', \mathbb{F}')$ -(strong) predictable representation property  $((Q', \mathbb{F}')\text{-p.r.p.})$  when any real  $(Q', \mathbb{F}')$ -local martingale can be written as  $m + \boldsymbol{\xi} \bullet \mathbf{X}'$  where  $m$  is a random variable  $\mathcal{F}'_0$ -measurable,  $\boldsymbol{\xi} = (\xi^1, \dots, \xi^l)$  is an  $\mathbb{F}'$ -predictable process and  $\boldsymbol{\xi} \bullet \mathbf{X}'$  is the vector stochastic integral (see [7] and [30]). As well known, this property is equivalent to the existence of a unique, modulo  $\mathcal{F}'_0$ , equivalent local martingale measure for  $\mathbf{X}'$  (see Proposition 3.1 in [2]). A particular case is when the local martingale measure for  $\mathbf{X}'$  is the unique equivalent martingale measure. In this case  $\mathcal{F}'_0$  is the trivial  $\sigma$ -algebra (see Theorem 11.2 in [18]).

In [6], given a real valued  $\mathbb{F}$ -semi-martingale  $X$  on a probability space  $(\Omega, \mathcal{F}, P)$  when there exists a unique equivalent martingale measure  $P^X$ , we studied the problem of the stability of the p.r.p. of  $X$  under enlargement of the reference filtration  $\mathbb{F}$ . In particular we assumed the existence of a second real semi-martingale  $Y$  on  $(\Omega, \mathcal{F}, P)$  endowed with a different reference filtration,  $\mathbb{H}$ , and admitting itself a unique martingale measure  $P^Y$ . We denoted by  $\mathbb{G}$  the filtration obtained by the union of  $\mathbb{F}$  and  $\mathbb{H}$  and we stated a representation theorem for the elements of  $\mathcal{M}^2(P, \mathbb{G})$ , the space of the real square-integrable  $(P, \mathbb{G})$ -martingales (see part ii) of Theorem 4.11 in [6]). More precisely we assumed the  $(P, \mathbb{G})$ -strong orthogonality of the martingale parts  $M$  and  $N$  of  $X$  and  $Y$  respectively, and we showed that every martingale in  $\mathcal{M}^2(P, \mathbb{G})$  can be uniquely represented as sum of an integral with respect to  $M$ , an integral with respect to  $N$  and an integral with respect to their quadratic covariation  $[M, N]$ . Equivalently we identified  $(M, N, [M, N])$  as a  $(P, \mathbb{G})$ -basis of real strongly orthogonal martingales (see, e. g. [10]). We stress that  $\mathbb{F}$  and  $\mathbb{H}$  could be taken larger than the natural filtration of  $X$  and  $Y$  respectively.

In this paper we deal with the multidimensional version of the representation theorem in [6]. Here  $\mathbf{X}$  is a  $(P, \mathbb{F})$  semi-martingale on  $(\Omega, \mathcal{F}, P)$  with values in  $\mathbb{R}^m$  and martingale part  $\mathbf{M}$ ,  $\mathbf{Y}$  is a  $(P, \mathbb{H})$  semi-martingale on  $(\Omega, \mathcal{F}, P)$  with values in  $\mathbb{R}^n$  and martingale part  $\mathbf{N}$ , the initial  $\sigma$ -algebras  $\mathcal{F}_0$  and  $\mathcal{H}_0$  are trivial,  $\mathbf{X}$  enjoys the  $(P, \mathbb{F})$ -p.r.p. and  $\mathbf{Y}$  enjoys the  $(P, \mathbb{H})$ -p.r.p. Finally, for all  $i = 1, \dots, m$ ,  $j = 1, \dots, n$ , we assume the  $(P, \mathbb{G})$ -strong orthogonality of the  $i$ -component of  $\mathbf{M}$ ,  $M^i$ , and of the  $j$ -component of  $\mathbf{N}$ ,  $N^j$ .

Our main result is that  $\mathcal{M}^2(P, \mathbb{G})$  coincides with the direct sum of three stable spaces of square-integrable martingales: the stable space generated by  $\mathbf{M}$ , the stable space generated by  $\mathbf{N}$  and the stable space generated by the family of processes  $([M^i, N^j], i = 1, \dots, m, j = 1, \dots, n)$  (see [18] for the theory of stable spaces generated by families of martingales and in particular page 114 for their definition and Theorem 4.60 at page 143 for their identification as space of vector integrals). More precisely any martingale in  $\mathcal{M}^2(P, \mathbb{G})$  can be uniquely represented as sum of elements of those stable spaces and every pair of elements of any two of those spaces is a pair of real  $(P, \mathbb{G})$ -strongly orthogonal martingales. In analogy with the unidimensional case, we can express the result saying that the triplet of vector processes given by  $\mathbf{M}$ ,  $\mathbf{N}$  and any vector martingale obtained by ordering the family  $([M^i, N^j], i = 1, \dots, m, j = 1, \dots, n)$  is a  $(P, \mathbb{G})$ -basis of multidimensional martingales.

Let us present the basic idea and the tools which we use here.

In order to get the heuristic of the result, but just for this, it helps to start reasoning with the particular case when  $\mathbf{X}$  and  $\mathbf{Y}$  coincide with their martingale parts  $\mathbf{M}$  and  $\mathbf{N}$ , respectively, and both  $\mathbf{M}$  and  $\mathbf{N}$  have strongly orthogonal components. In fact last assumption implies that under  $P$  all vector stochastic integrals with respect to  $\mathbf{M}$  are componentwise stochastic integrals, that is

$$\boldsymbol{\xi} \bullet \mathbf{M} = \sum_{i=1}^m \int_0^\cdot \xi_t^i dM_t^i$$

(see [7] and Theorem 1.17 in [30]). Obviously the same holds for  $\mathbf{N}$ . So that applying Ito's Lemma to the the product of a  $(P, \mathbb{F})$ -martingale with a  $(P, \mathbb{H})$ -martingale and, taking into account the p.r.p. of  $\mathbf{M}$  and  $\mathbf{N}$ , we realize that for representing all  $(P, \mathbb{G})$ -martingales we need the family of processes  $([M^i, N^j] \ i = 1, \dots, m, \ j = 1, \dots, n)$  in addition to  $\mathbf{M}$  and  $\mathbf{N}$ .

The proof of our main result when  $\mathbf{X} \equiv \mathbf{M}$  and  $\mathbf{Y} \equiv \mathbf{N}$ , not necessarily with pairwise strongly orthogonal components, is based on two statements of the theory of stable spaces generated by multidimensional square integrable martingales, or equivalently of the theory of vector stochastic integrals with respect to square-integrable martingales. The two statements can be roughly resumed as follows. Given two square integrable martingales with mutually pairwise strongly orthogonal components, the stable space generated by one of them is contained in the subspace orthogonal to the other. If a multidimensional martingale can be decomposed in packets of components mutually pairwise strongly orthogonal, then the stable space generated by this martingale is the direct sum of the stable spaces generated by the "packets martingales". (see Lemma 3.1 and Remark (3.2)).

In our framework the assumption that, for all  $i = 1, \dots, m$  and  $j = 1, \dots, n$ ,  $M^i$  and  $N^j$  are  $(P, \mathbb{G})$ -strongly orthogonal martingales together with the  $(P, \mathbb{F})$ -p.r.p. of  $\mathbf{M}$  and the  $(P, \mathbb{H})$ -p.r.p. of  $\mathbf{N}$ , allows to show the  $P$ -independence of  $\mathbb{F}$  and  $\mathbb{H}$ . Using this fact we can prove that, for all  $i = 1, \dots, m$  and  $j = 1, \dots, n$ ,  $[M^i, N^j]$  is  $(P, \mathbb{G})$ -strongly orthogonal to  $M^h$ ,  $h = 1, \dots, m$ , and to  $N^k$ ,  $k = 1, \dots, n$ . Then the first general statement above implies that the stable space generated by  $([M^i, N^j] \ i = 1, \dots, m, \ j = 1, \dots, n)$  is contained in the orthogonal of the subspace generated by  $(\mathbf{M}, \mathbf{N})$ , that is any element of the stable subspace generated by  $([M^i, N^j] \ i = 1, \dots, m, \ j = 1, \dots, n)$  is a real martingale  $(P, \mathbb{G})$ -strongly orthogonal to any element of the stable space generated by  $(\mathbf{M}, \mathbf{N})$ . At the same time using the  $(P, \mathbb{G})$ -strong orthogonality of  $M^i$  and  $N^j$ , for all  $i = 1, \dots, m$  and  $j = 1, \dots, n$ , we are able to show that  $P$  is the unique equivalent martingale measure On  $(\Omega, \mathcal{G}_T)$  for the vector processes  $\mathbf{M}$ ,  $\mathbf{N}$  and the real valued processes  $[M^i, N^j]$ ,  $i = 1, \dots, m, \ j = 1, \dots, n$ . Then the result follows by the second general statement above.

The well-known formula

$$[M^i, N^j]_t = \langle M^{c,i}, N^{c,j} \rangle_t + \sum_{s \leq t} \Delta M_s^i \Delta N_s^j, \quad (1)$$

where  $M^{c,i}$  and  $N^{c,j}$  are the continuous parts of  $M^i$  and  $N^j$  respectively, has two immediate consequences. When  $\mathbf{M}$  and  $\mathbf{N}$  have continuous trajectories, thanks to the  $P$ -independence of  $\mathbb{F}$  and  $\mathbb{H}$ ,  $\mathbf{M}$  and  $\mathbf{N}$  are enough to represent every  $(P, \mathbb{G})$ -martingale and

the same happens when  $\mathbf{M}$  and  $\mathbf{N}$  admit only totally inaccessible jump times. Therefore, in particular if  $\mathbf{M}$  and  $\mathbf{N}$  are quasi-left continuous martingales then  $\mathbf{M}$  and  $\mathbf{N}$  are a  $(P, \mathbb{G})$ -basis of multidimensional martingales for  $\mathcal{M}^2(P, \mathbb{G})$ . Instead, when  $\mathbf{M}$  and  $\mathbf{N}$  jump simultaneously at accessible jump times, the stable space generated by the covariation terms has to be added in order to get the representation of  $\mathcal{M}^2(P, \mathbb{G})$ .

When  $\mathbf{X}$  and  $\mathbf{Y}$  are not trivial semi-martingales, that is when they do not coincide with  $\mathbf{M}$  and  $\mathbf{N}$  respectively, under suitable assumptions, we are able to prove that the representation is the same. We stress that the approach is slightly different from that used to handle the unidimensional case (see Section 4.2 in [6]) and also that the hypotheses are simpler than those required in that paper. Here the question reduces to ask for conditions under which the  $(P^X, \mathbb{F})$ -p.r.p. of  $\mathbf{X}$  and the  $(P^Y, \mathbb{H})$ -p.r.p. of  $\mathbf{Y}$  are equivalent to the  $(P, \mathbb{F})$ -p.r.p. of  $\mathbf{M}$  and the  $(P, \mathbb{H})$ -p.r.p. of  $\mathbf{N}$  respectively, that is to look for conditions under which the *invariance of the p.r.p. under equivalent changes of probability measure* holds (see, e.g. Lemma 2.5 in [23]). Clearly the involved changes of measure are two: one of them makes  $\mathbf{X}$  an  $\mathbb{F}$ -martingale, the other one makes  $\mathbf{Y}$  an  $\mathbb{H}$ -martingale. Our key assumption is the local square-integrability of the corresponding Girsanov's derivatives. This assumption provides in particular the *structure condition* for  $\mathbf{X}$  and  $\mathbf{Y}$ . It is also to note that in our previous paper we assumed a bound on the jumps size of  $\mathbf{M}$  and  $\mathbf{N}$ , which now follows as a consequence (see **H4**) in [6]).

Moreover here, like in [6], we obtain also a second representation result:  $\mathbf{X}$ ,  $\mathbf{Y}$  and any ordering of the family  $([X^i, Y^j], i = 1, \dots, m, j = 1, \dots, n)$  form a basis of multidimensional martingales for the  $\mathbb{G}$ -square-integrable martingales under a new probability measure on  $\mathcal{G}_T$ ,  $Q$ , equivalent to  $P|_{\mathcal{G}_T}$ . Indeed, since  $\mathbf{M}$  and  $\mathbf{N}$  enjoy  $(P, \mathbb{F})$ -p.r.p. and  $(P, \mathbb{H})$ -p.r.p. respectively, the assumed  $(P, \mathbb{G})$ -strong orthogonality of  $M^i$  and  $N^j$ , for all  $i = 1, \dots, m$  and  $j = 1, \dots, n$ , implies the  $P$ -independence of  $\mathbb{F}$  and  $\mathbb{H}$ . This allows us to construct a *decoupling probability measure* for  $\mathbb{F}$  and  $\mathbb{H}$  on  $\mathcal{G}_T$ ,  $Q$ , equivalent to  $P$  and such that  $\mathbf{X}$  enjoys the  $(Q, \mathbb{F})$ -p.r.p. and  $\mathbf{Y}$  enjoys the  $(Q, \mathbb{H})$ -p.r.p.. Obviously, the  $Q$ -independence of  $\mathbb{F}$  and  $\mathbb{H}$  implies  $(Q, \mathbb{G})$ -strong orthogonality of  $X^i - X_0^i$  and  $Y^j - Y_0^j$ , for all  $i = 1, \dots, m, j = 1, \dots, n$  and this allows to obtain the announced representation.

The first application answers to the following question. Let a filtration  $\mathbb{G}$  be obtained by the union of a finite family of filtrations,  $(\mathbb{F}^1, \dots, \mathbb{F}^d)$ , such that  $\mathbb{F}^i$ ,  $i = 1, \dots, d$ , is the reference filtration of a real martingale  $M^i$ . Let us assume that  $M^i$  enjoys the  $(P, \mathbb{F}^i)$ -p.r.p.. Can we determine a  $(P, \mathbb{G})$ -basis of real martingales? Here we give conditions that make independent the filtrations  $\mathbb{F}^1, \dots, \mathbb{F}^d$  and as a consequence allow to identify a  $(P, \mathbb{G})$ -basis. We also underline the link with the notion of multiplicity of a filtration (see [11]).

The second application falls into the framework of mathematical finance and more precisely in the reduced form approach of credit risk modeling. In [5], under completeness of the reference market and *density hypothesis* for the default time, the authors got the martingale representation on the full market under the historical measure (see also [21] for a similar result). Our theorem provides a new proof of that result under slightly different hypotheses. Indeed we allow the risky asset price to be a multidimensional semi-martingale,  $\mathbf{S}$  and we assume the *immersion property* under the historical measure  $P$  of the market

filtration  $\mathbb{F}$  into the filtration  $\mathbb{G}$  defined at time  $t$  by

$$\mathcal{G}_t := \cap_{s>t} \mathcal{F}_s \vee \sigma(\tau \wedge s).$$

One of the key points of our result is, like in [5], the existence of a decoupling measure. This measure preserves either the law of  $S$  and in particular of its martingale part  $M$ , or the law of the *compensated default process*  $H$  defined at time  $t$  by

$$H_t := \mathbb{I}_{\{\tau \leq t\}} - \int_0^{\tau \wedge t} \frac{dF_u}{1 - F_u},$$

where  $F$  denotes the continuous distribution function of  $\tau$ . The last fact joint with some technical conditions implies that  $M$  and  $H$  enjoy the p.r.p. under the decoupling measure as well under  $P$ . From our theorem we immediately derive that  $M$  and  $H$  are a basis of multidimensional martingales for the filtration  $\mathbb{G}$  under the decoupling measure. In fact by the density hypothesis the default time does not coincide with any jump time of the asset price with positive probability. Then the invariance property of the p.r.p. under equivalent changes of measure allows to establish the representation.

This note is organized as follows. In Section 2, we introduce the notations and some basic definitions, we state the hypotheses, we discuss their consequences and we derive a fundamental ingredient for our result, that is the p.r.p. under  $P$  for  $\mathbf{M}$  and  $\mathbf{N}$  with respect to  $\mathbb{F}$  and  $\mathbb{H}$  respectively. Section 3 is devoted to the main result. Section 4 contains the applications.

## 2 Setting and hypotheses

Let us fix some notations used in all the paper.

Let  $T$  be a finite horizon. Let  $\mathbf{S} = (\mathbf{S}_t)_{t \in [0, T]} = ((S_t^1, \dots, S_t^l))_{t \in [0, T]}$  be a càdlàg square-integrable  $l$ -dimensional semi-martingale on a filtered probability space,  $(\Omega, \mathcal{A}, \mathbb{A}, R)$ , with  $\mathbb{A} = (\mathcal{A}_{t \in [0, T]})$  under usual conditions. We will denote by  $\mathbb{P}(\mathbf{S}, \mathbb{A})$  the set of martingale measures for  $\mathbf{S}$  on  $(\Omega, \mathcal{A}_T)$  equivalent to  $R|_{\mathcal{A}_T}$ .

Let  $Q \in \mathbb{P}(\mathbf{S}, \mathbb{A})$ . We will denote by  $\mathcal{L}^2(\mathbf{S}, Q, \mathbb{A})$  the set of the  $\mathbb{A}$ -predictable  $l$ -dimensional processes  $\boldsymbol{\xi} = (\boldsymbol{\xi}_t)_{t \in [0, T]} = ((\xi_t^1, \dots, \xi_t^l))_{t \in [0, T]}$  such that

$$E^Q \left[ \int_0^T \boldsymbol{\xi}_t^{tr} C_t^{\mathbf{S}} \boldsymbol{\xi}_t dB_t^{\mathbf{S}} \right] < +\infty,$$

where

$$B_t^{\mathbf{S}} := \sum_{i=1}^l \langle S^i \rangle_t^{Q, \mathbb{A}}, \quad c_{ij}^{\mathbf{S}}(t) := \frac{d\langle S^i, S^j \rangle_t^{Q, \mathbb{A}}}{dB_t^{\mathbf{S}}}, \quad i, j \in (1, \dots, l) \quad (2)$$

with  $[S^i, S^j]$  and  $\langle S^i, S^j \rangle^{Q, \mathbb{A}}$ ,  $i, j \in (1, \dots, l)$  the quadratic covariation process and the sharp covariation process of  $S^i$  and  $S^j$  respectively.

Following [30] we will endow  $\mathcal{L}^2(\mathbf{S}, Q, \mathbb{A})$  with the norm

$$\|\boldsymbol{\xi}\|_{\mathcal{L}^2(\mathbf{S}, Q, \mathbb{A})}^2 := E^Q \left[ \int_0^T \boldsymbol{\xi}_t^{tr} C_t^{\mathbf{S}} \boldsymbol{\xi}_t dB_t^{\mathbf{S}} \right]. \quad (3)$$

It is possible to prove that

$$\|\boldsymbol{\xi}\|_{\mathcal{L}^2(\mathbf{S}, Q, \mathbb{A})}^2 = E^Q[\boldsymbol{\xi} \bullet \mathbf{S}]_T \quad (4)$$

(see formula (3.5) in [30]) where  $\boldsymbol{\xi} \bullet \mathbf{S}$  denotes the vector stochastic integral of  $\boldsymbol{\xi} \in \mathcal{L}^2(\mathbf{S}, Q, \mathbb{A})$  with respect to  $\mathbf{S}$ .

We recall that  $\boldsymbol{\xi} \bullet \mathbf{S}$  is a one-dimensional process and therefore different from the vector  $(\int_0^\cdot \xi_t^1 dS_t^1, \dots, \int_0^\cdot \xi_t^l dS_t^l)$ . Moreover  $\boldsymbol{\xi} \bullet \mathbf{S}$  coincides with  $\sum_{i=1}^l \int_0^\cdot \xi_t^i dS_t^i$ , when  $\mathbf{S}$  has pairwise  $(Q, \mathbb{A})$ -strongly orthogonal components (see [7] and [30]). As noted by Chernyi and Shiryaev, unless the construction of the vector stochastic integral is a bit complicated, this notion provides the closeness of the space of stochastic integrals. The notion of componentwise stochastic integral in general does not (for a detailed discussion see [30]).

We will set

$$K^2(\Omega, \mathbb{A}, Q, \mathbf{S}) := \{(\boldsymbol{\xi} \bullet \mathbf{S})_T, \boldsymbol{\xi} \in \mathcal{L}^2(\mathbf{S}, Q, \mathbb{A})\}.$$

We recall that, when  $\mathcal{A}_0$  is trivial,  $\mathbb{P}(\mathbf{S}, \mathbb{A})$  is a singleton, more precisely

$$\mathbb{P}(\mathbf{S}, \mathbb{A}) = \{P^{\mathbf{S}}\},$$

if and only if  $\mathbf{S}$  enjoys the  $(P^{\mathbf{S}}, \mathbb{A})$ -p.r.p. that is if and only if each  $H$  in  $L^2(\Omega, \mathcal{A}_T, P^{\mathbf{S}})$  can be represented  $P^{\mathbf{S}}$ -a.s. , up to an additive constant, as vector stochastic integral with respect to  $\mathbf{S}$  that is

$$H = H_0 + (\boldsymbol{\xi}^H \bullet \mathbf{S})_T,$$

with  $H_0$  a constant and  $\boldsymbol{\xi}^H \in \mathcal{L}^2(\mathbf{S}, P^{\mathbf{S}}, \mathbb{A})$ . More briefly,  $\mathbb{P}(\mathbf{S}, \mathbb{A}) = \{P^{\mathbf{S}}\}$  if and only if

$$L_0^2(\Omega, \mathcal{A}_T, P^{\mathbf{S}}) = K^2(\Omega, \mathbb{A}, P^{\mathbf{S}}, \mathbf{S}), \quad (5)$$

where  $L_0^2(\Omega, \mathcal{A}_T, P^{\mathbf{S}})$  is the set of all real centered  $P^{\mathbf{S}}$ -square integrable  $\mathcal{A}_T$ -measurable random variables (see [17]).

We will indicate by  $\mathcal{M}^2(R, \mathbb{A})$  the set of all real square integrable  $(R, \mathbb{A})$ -martingales on  $[0, T]$ , which is a Banach space with the norm

$$\|S\|_{\mathcal{M}^2(R, \mathbb{A})}^2 = E^R[S_T^2] \quad (6)$$

(see [27])<sup>1</sup>.

Finally following [18] we will denote by  $\mathcal{Z}^2(\boldsymbol{\mu})$  the stable space generated by a finite set of square-integrable martingales  $\boldsymbol{\mu} \subset \mathcal{M}^2(R, \mathbb{A})$ . The general element of  $\mathcal{Z}^2(\boldsymbol{\mu})$  is a vector stochastic integral with respect to  $\boldsymbol{\mu}$  (see Theorem 4.60 page 143 in [18]).

Now let us introduce the general setup of our result.

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<sup>1</sup> by Jensen inequality  $S^2$  is as a sub-martingale so that  $\sup_{t \leq T} E[S_t^2] \leq E[S_T^2] < +\infty$  and by Doob's inequality  $E[\sup_{t \leq T} S_t^2] < +\infty$ . An equivalent norm on  $\mathcal{M}^2(R, \mathbb{A})$  is

$$E[\sup_{t \leq T} S_t^2]$$

(see pages 26, 27 in [18])

Given a probability space  $(\Omega, \mathcal{F}, P)$ , a finite time horizon  $T \in (0, +\infty)$  and two filtrations  $\mathbb{F}$  and  $\mathbb{H}$  under standard conditions and with  $\mathcal{F}_T \subset \mathcal{F}$  and  $\mathcal{H}_T \subset \mathcal{F}$ , we consider two square-integrable semi-martingales and more precisely an  $m$ -dimensional  $(P, \mathbb{F})$ -semi-martingale  $X$  and an  $n$ -dimensional  $(P, \mathbb{H})$ -semi-martingale  $Y$  with canonical decomposition

$$\mathbf{X} = \mathbf{X}_0 + \mathbf{M} + \mathbf{A}, \quad \mathbf{Y} = \mathbf{Y}_0 + \mathbf{N} + \mathbf{D} \quad (7)$$

such that

$$E^P \left[ \|\mathbf{X}_0\|_{\mathbb{R}^m}^2 + \sum_{i,j=1}^m [M^i, M^j]_T + \sum_{i=1}^m |A^i|_T^2 \right] < +\infty \quad (8)$$

and

$$E^P \left[ \|\mathbf{Y}_0\|_{\mathbb{R}^n}^2 + \sum_{i,j=1}^n [N^i, N^j]_T + \sum_{i=1}^n |D^i|_T^2 \right] < +\infty. \quad (9)$$

Here  $\mathbf{M} = (M^1, \dots, M^m)$  is an  $m$ -dimensional  $(P, \mathbb{F})$ -martingale with  $M^i \in \mathcal{M}^2(P, \mathbb{F})$  for all  $i = 1, \dots, m$ ,  $\mathbf{A} = (A^1, \dots, A^m)$  is an  $m$ -dimensional  $\mathbb{F}$ -predictable process of finite variation,  $\mathbf{M}_0 = \mathbf{A}_0 = 0$  and  $|A^i|$  denotes the total variation process of  $A^i$ . Note that by the integrability condition (8) it follows

$$E^P \left[ \sup_{t \in [0, T]} \|\mathbf{X}_t\|_{\mathbb{R}^m}^2 \right] < +\infty. \quad (10)$$

Analogous considerations hold for  $\mathbf{Y}$ .

We assume that the sets  $\mathbb{P}(\mathbf{X}, \mathbb{F})$  and  $\mathbb{P}(\mathbf{Y}, \mathbb{H})$  are singletons and more precisely

**A1)**  $\mathbb{P}(\mathbf{X}, \mathbb{F}) = \{P^{\mathbf{X}}\}$ ,  $\mathbb{P}(\mathbf{Y}, \mathbb{H}) = \{P^{\mathbf{Y}}\}$ .

We introduce the Radon-Nikodym derivatives

$$L_t^{\mathbf{X}} := \frac{dP^{\mathbf{X}}}{dP|_{\mathcal{F}_t}}, \quad L_t^{\mathbf{Y}} := \frac{dP^{\mathbf{Y}}}{dP|_{\mathcal{H}_t}}$$

and their inverses

$$\tilde{L}_t^{\mathbf{X}} := \frac{1}{L_t^{\mathbf{X}}} = \frac{dP|_{\mathcal{F}_t}}{dP^{\mathbf{X}}}, \quad \tilde{L}_t^{\mathbf{Y}} := \frac{1}{L_t^{\mathbf{Y}}} = \frac{dP|_{\mathcal{H}_t}}{dP^{\mathbf{Y}}}.$$

Then we require the following regularity conditions on them

**A2)**

$$L_T^{\mathbf{X}} \in L_{loc}^2(\Omega, \mathcal{F}_T, P), \quad L_T^{\mathbf{Y}} \in L_{loc}^2(\Omega, \mathcal{H}_T, P). \quad (11)$$

Let us now discuss the consequences of our assumptions.

Hypothesis **A1)** implies that  $\mathcal{F}_0$  and  $\mathcal{H}_0$  are trivial and that  $\mathbf{X}$  enjoys the  $(P^{\mathbf{X}}, \mathbb{F})$ -p.r.p. and  $\mathbf{Y}$  enjoys the  $(P^{\mathbf{Y}}, \mathbb{H})$ -p.r.p. (see [17]).

Using Hypothesis **A2)** we derive the structure condition for  $\mathbf{X}$  and  $\mathbf{Y}$  and in particular



the existence of an  $\mathbb{F}$ -predictable  $m$ -dimensional process  $\boldsymbol{\alpha} = (\alpha_t^1, \dots, \alpha_t^m)_{t \in [0, T]}$  and an  $\mathbb{H}$ -predictable process  $\boldsymbol{\delta} = (\delta_t^1, \dots, \delta_t^n)_{t \in [0, T]}$  such that for all  $i \in (1, \dots, m)$  and  $j \in (1, \dots, n)$

$$A_t^i = \int_0^t \alpha_s^i d\langle M^i \rangle_s^{P, \mathbb{F}}, \quad D_t^j = \int_0^t \delta_s^j d\langle N^j \rangle_s^{P, \mathbb{H}} \quad (12)$$

(see e.g. Definition 1.1 and Theorem 2.2. in [8]).

Moreover it holds

$$\alpha^i \in L_{loc}^2(P \times d\langle M^i \rangle^{P, \mathbb{F}}), \quad \delta^j \in L_{loc}^2(P \times d\langle N^j \rangle^{P, \mathbb{H}})$$

that is, following the notations in [18],  $\alpha^i \in L_{loc}^2(M^i)$  and  $\delta^j \in L_{loc}^2(N^j)$  (see Proposition 4 in [28] or Theorem 1 in [29]).

Moreover the square-integrability of  $\mathbf{X}$  and  $\mathbf{Y}$  together with assumption **A2**) imply that for all  $i = 1, \dots, m$  and for all  $j = 1, \dots, n$ , for all  $t \in [0, T]$

$$\langle \tilde{L}_t^{\mathbf{X}}, X_t^i \rangle^{P^{\mathbf{X}}, \mathbb{F}}, \quad \langle \tilde{L}_t^{\mathbf{Y}}, Y_t^j \rangle^{P^{\mathbf{Y}}, \mathbb{H}} \quad (13)$$

exist (see e.g. VII 39 in [13]).<sup>2</sup>

Finally **A1**) and **A2**) allow to transfer the p.r.p. from  $\mathbf{X}$  to its martingale part,  $\mathbf{M}$ , and from  $\mathbf{Y}$  to its martingale part,  $\mathbf{N}$ . Indeed we can announce the following result.

**Proposition 2.1.** *Let **A1**) and **A2**) be verified. Then  $\mathbf{M}$  enjoys the  $(P, \mathbb{F})$ -p.r.p..*

*Proof.* Set

$$\tilde{X}_t^i = X_t^i - \int_0^t \frac{1}{\tilde{L}_{s-}^{\mathbf{X}}} d\langle \tilde{L}^{\mathbf{X}}, X^i \rangle_s^{P^{\mathbf{X}}, \mathbb{F}}, \quad i = 1, \dots, m.$$

Then by Lemma 2.4 in [23] the process  $\tilde{\mathbf{X}} = (\tilde{X}_t^1, \dots, \tilde{X}_t^m)_{t \in [0, T]}$  is a  $(P, \mathbb{F})$ -local martingale which enjoys the  $(P, \mathbb{F})$ -p.r.p.. Moreover it coincides with  $\mathbf{M}$ . In fact, fixed  $i \in (1, \dots, m)$ ,

$$\tilde{X}_t^i - M_t^i = X_0^i + \int_0^t \alpha_s^i d\langle M^i \rangle_s^{P, \mathbb{F}} - \int_0^t \frac{1}{\tilde{L}_{s-}^{\mathbf{X}}} d\langle \tilde{L}^{\mathbf{X}}, X^i \rangle_s^{P^{\mathbf{X}}, \mathbb{F}}$$

is a predictable  $(\mathbb{F}, P)$ -local martingale, so it has to be continuous (see Theorem 43 Chapter IV in [25]). Moreover it has finite variation, so that it is necessarily null.  $\square$

As in formula (2), for any fixed  $t \in [0, T]$ , consider the matrix  $C_t^{\mathbf{M}}$  with generic element defined by

$$c_{ij}^{\mathbf{M}}(t) := \frac{d\langle M^i, M^j \rangle_t^{P, \mathbb{F}}}{dB_t^{\mathbf{M}}}$$

with

$$B_t^{\mathbf{M}} := \sum_{i=1}^m \langle M^i \rangle_t^{P, \mathbb{F}}.$$

---

<sup>2</sup> from (10) it follows that  $\sup_{t \in [0, T]} \|\mathbf{X}_t\|_{\mathbb{R}^m} < +\infty$ ,  $P$ -a.s. and  $P^X$ -a.s.. Therefore, for any divergent sequence  $\{c_n\}_{n \in \mathbb{N}}$  of real number, if

$$T_n := \inf \left\{ t \in [0, T] : \sup_{s \leq t} \|\mathbf{X}_s\|_{\mathbb{R}^m} > c_n \right\},$$

then  $\{T_n\}_{n \in \mathbb{N}}$  is a divergent sequence of stopping times such that  $\sup_{t \in [0, T]} \|\mathbf{X}_{T_n \wedge t}\|_{\mathbb{R}^m} < c_n$ , that is  $\mathbf{X}$  is locally bounded



**Corollary 2.2.** *Let  $\hat{\lambda}$  be defined by*

$$C_t^{\mathbf{M}} \hat{\lambda}_t := \gamma_t, \quad 0 \leq t \leq T$$

where

$$\gamma_t^i := \alpha_t^i c_{ii}^M(t).$$

Then

$$i) \quad \hat{\lambda}^{\text{tr}} \Delta \mathbf{M} < 1;$$

ii)  $P^{\mathbf{X}}$  coincides with the minimal martingale measure for  $\mathbf{X}$ .

*Proof.* By hypotheses **A1**) and **A2**)  $L^{\mathbf{X}}$  is a locally square integrable strict martingale density under  $P$ , that is  $L^{\mathbf{X}} \in \mathcal{M}_{loc}^2(P, \mathbb{F})$ , so that for all  $t \in [0, T]$

$$L_t^{\mathbf{X}} = \mathcal{E} \left( -(\hat{\lambda} \bullet \mathbf{M})_t + V_t \right)$$

or equivalently  $L^{\mathbf{X}}$  solves the equation

$$Z = 1 - Z \hat{\lambda} \bullet \mathbf{M} + V,$$

where  $V$  is a  $(P, \mathbb{F})$ -local martingale with real values, null at zero and  $(P, \mathbb{F})$ -strongly orthogonal to  $M^i$  for each  $i \in (1, \dots, m)$  (see Theorem 2.2. in [8] or Theorem 1 in [29]). Previous proposition forces  $V$  to be null so that for all  $t \in [0, T]$

$$L_t^{\mathbf{X}} = \exp \left( -(\hat{\lambda} \bullet \mathbf{M})_t - \sum_{i,j=1}^m \int_0^t \hat{\lambda}_s^i \hat{\lambda}_s^j d\langle M^{c,i}, M^{c,j} \rangle_s \right) \prod_{0 \leq s \leq t} (1 - \hat{\lambda}_s^{\text{tr}} \Delta \mathbf{M}_s) e^{\hat{\lambda}_s^{\text{tr}} \Delta \mathbf{M}_s}$$

where  $\mathbf{M}^c = (M^{c,1}, \dots, M^{c,m})$  is the continuous martingale part of  $\mathbf{M}$ . Since  $L^{\mathbf{X}}$  by assumption is the derivative of an equivalent change of measure, then it has to be strictly positive so that part i) follows.

Proposition 3.1 in [4] proves part ii). □

**Remark 2.3.** *We recall that condition  $\hat{\lambda}^{\text{tr}} \Delta \mathbf{M} < 1$  doesn't follow by the existence of a strict martingale density (see Section 4 in [3] for a counter-example in the one-dimensional case, where the condition turns into  $\alpha \Delta M < 1$ ). Only when  $P(\mathbf{X}, \mathbb{F})$  is a singleton the condition necessarily follows.*

### 3 Two bases of martingales

In this section we present the multidimensional version of Theorem 4.11 in [6]. The key assumption is

**A3)** for any  $i \in (1, \dots, m)$  and  $j \in (1, \dots, n)$ ,  $M^i$  and  $N^j$  are real  $(P, \mathbb{G})$ -strongly orthogonal martingales, where

$$\mathbb{G} := \mathbb{F} \vee \mathbb{H},$$

or equivalently for any  $i \in (1, \dots, m)$  and  $j \in (1, \dots, n)$ , the process  $[M^i, N^j]$  is a real uniformly integrable  $(P, \mathbb{G})$ -martingale with null initial value.

Let us denote by  $[\mathbf{M}, \mathbf{N}]^{\mathbf{V}}$  the process

$$([M^1, N^1], \dots, [M^1, N^n], [M^2, N^1], \dots, [M^2, N^n], \dots, [M^m, N^1], \dots, [M^m, N^n]).$$

Then, under assumption **A3**,  $[\mathbf{M}, \mathbf{N}]^{\mathbf{V}}$  is a  $(P, \mathbb{G})$ -martingale with values in  $\mathbb{R}^{mn}$ .

Before announcing the main theorem we state a general result.

**Lemma 3.1.** *On a filtered probability space  $(\Omega, \mathcal{A}, \mathbb{A}, R)$  let consider two processes  $\boldsymbol{\mu} = (\mu^1, \dots, \mu^r)$  and  $\boldsymbol{\mu}' = (\mu'^1, \dots, \mu'^s)$  such that, for all  $i = 1, \dots, r$  and  $j = 1, \dots, s$ ,  $\mu^i$  and  $\mu'^j$  are  $(R, \mathbb{A})$ -strongly orthogonal real martingales in  $\mathcal{M}^2(R, \mathbb{A})$ . Then*

*i) for any  $\boldsymbol{\xi} \in \mathcal{L}^2(\boldsymbol{\mu}, R, \mathbb{A})$  and  $\boldsymbol{\eta} \in \mathcal{L}^2(\boldsymbol{\mu}', R, \mathbb{A})$ , the processes  $\boldsymbol{\xi} \bullet \boldsymbol{\mu}$  and  $\boldsymbol{\eta} \bullet \boldsymbol{\mu}'$ , are real  $(R, \mathbb{A})$ -strongly orthogonal martingales, that is  $\boldsymbol{\xi} \bullet \boldsymbol{\mu} \cdot \boldsymbol{\eta} \bullet \boldsymbol{\mu}'$  is a real uniformly integrable  $(R, \mathbb{A})$ -martingale with null initial value.*

*ii) Moreover let  $\boldsymbol{\mu}'' = (\mu''^1, \dots, \mu''^w)$  be a  $(R, \mathbb{A})$ -martingale such that, for all fixed  $h \in (1, \dots, w)$ , either the real processes  $\mu''^h$  and  $\mu^i$ , for every  $i = 1, \dots, r$ , or the real processes  $\mu''^h$  and  $\mu'^j$ , for every  $j = 1, \dots, s$ , are  $(R, \mathbb{A})$ -strongly orthogonal martingales in  $\mathcal{M}^2(R, \mathbb{A})$ .*

*Then for all  $\boldsymbol{\Theta} \in \mathcal{L}^2((\boldsymbol{\mu}, \boldsymbol{\mu}', \boldsymbol{\mu}''), R, \mathbb{A})$  there exists a unique triplet  $\boldsymbol{\xi}, \boldsymbol{\xi}', \boldsymbol{\xi}''$  with  $\boldsymbol{\xi} \in \mathcal{L}^2(\boldsymbol{\mu}, R, \mathbb{A})$ ,  $\boldsymbol{\xi}' \in \mathcal{L}^2(\boldsymbol{\mu}', R, \mathbb{A})$  and  $\boldsymbol{\xi}'' \in \mathcal{L}^2(\boldsymbol{\mu}'', R, \mathbb{A})$  such that*

$$\boldsymbol{\Theta} \bullet (\boldsymbol{\mu}, \boldsymbol{\mu}', \boldsymbol{\mu}'') = \boldsymbol{\xi} \bullet \boldsymbol{\mu} + \boldsymbol{\xi}' \bullet \boldsymbol{\mu}' + \boldsymbol{\xi}'' \bullet \boldsymbol{\mu}''.$$

*Proof.* As far as the first statement is concerned, fixed  $i \in (1, \dots, r)$ ,  $\mu^i$  is  $(R, \mathbb{A})$ -strongly orthogonal to all elements of  $\mathcal{Z}^2(\boldsymbol{\mu}')$  (see point a) of Theorem 4.7, page 116 in [18]), that is to every vector integral with respect to  $\boldsymbol{\mu}'$  of a process in  $\mathcal{L}^2(\boldsymbol{\mu}', R, \mathbb{A})$  (see Theorem 4.60 page 143 in [18]). Then, since  $i$  is arbitrary, by the same tools and a symmetric argument, the vector integral with respect to  $\boldsymbol{\mu}'$  of a fixed process in  $\mathcal{L}^2(\boldsymbol{\mu}', R, \mathbb{A})$  is  $(R, \mathbb{A})$ -orthogonal to the vector integral with respect to  $\boldsymbol{\mu}$  of a fixed process in  $\mathcal{L}^2(\boldsymbol{\mu}, R, \mathbb{A})$ .

The second statement follows by a slight generalization of Theorem 36, Chapter IV of [27], which implies the result when  $\boldsymbol{\mu}, \boldsymbol{\mu}'$  and  $\boldsymbol{\mu}''$  are real martingales.

Let  $\mathcal{I}$  be the space of processes

$$\mathbf{H} \bullet \boldsymbol{\mu} + \mathbf{H}' \bullet \boldsymbol{\mu}' + \mathbf{H}'' \bullet \boldsymbol{\mu}''$$

with  $\mathbf{H} \in \mathcal{L}^2(\boldsymbol{\mu}, R, \mathbb{A})$ ,  $\mathbf{H}' \in \mathcal{L}^2(\boldsymbol{\mu}', R, \mathbb{A})$ ,  $\mathbf{H}'' \in \mathcal{L}^2(\boldsymbol{\mu}'', R, \mathbb{A})$ . The stable space  $\mathcal{Z}^2(\boldsymbol{\mu}, \boldsymbol{\mu}', \boldsymbol{\mu}'')$  contains  $\mathcal{Z}^2(\boldsymbol{\mu})$ ,  $\mathcal{Z}^2(\boldsymbol{\mu}')$  and  $\mathcal{Z}^2(\boldsymbol{\mu}'')$  and therefore it contains  $\mathcal{I}$  (see Proposition 4.5 page 114 and Theorem 4.35 page 130 in [18]). Moreover  $\mathcal{I}$  turns out to be stable and then it coincides with  $\mathcal{Z}^2(\boldsymbol{\mu}, \boldsymbol{\mu}', \boldsymbol{\mu}'')^3$ .

First of all we show that  $\mathcal{I}$  is closed. To this end we consider the application which maps  $\mathcal{L}^2(\boldsymbol{\mu}, R, \mathbb{A}) \times \mathcal{L}^2(\boldsymbol{\mu}', R, \mathbb{A}) \times \mathcal{L}^2(\boldsymbol{\mu}'', R, \mathbb{A})$  into  $\mathcal{M}^2(R, \mathbb{A})$  in this way

$$(\mathbf{H}, \mathbf{H}', \mathbf{H}'') \rightarrow \mathbf{H} \bullet \boldsymbol{\mu} + \mathbf{H}' \bullet \boldsymbol{\mu}' + \mathbf{H}'' \bullet \boldsymbol{\mu}''.$$

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<sup>3</sup> Note that  $\mathcal{I}$  contains all the stochastic integrals with respect to any components of  $\boldsymbol{\mu}, \boldsymbol{\mu}'$  and  $\boldsymbol{\mu}''$

We prove that this application is an isometry. We then conclude the proof by observing that  $\mathcal{I}$  is the image through the above application of  $\mathcal{L}^2(\boldsymbol{\mu}, R, \mathbb{A}) \times \mathcal{L}^2(\boldsymbol{\mu}', R, \mathbb{A}) \times \mathcal{L}^2(\boldsymbol{\mu}'', R, \mathbb{A})$ , which is an Hilbert space with scalar product

$$\langle (\mathbf{H}, \mathbf{H}', \mathbf{H}''), (\mathbf{K}, \mathbf{K}', \mathbf{K}'') \rangle_{\mathcal{L}^2(\boldsymbol{\mu}, R, \mathbb{A}) \times \mathcal{L}^2(\boldsymbol{\mu}', R, \mathbb{A}) \times \mathcal{L}^2(\boldsymbol{\mu}'', R, \mathbb{A})} := E^R \left[ \int_0^T \mathbf{H}_t^{tr} C_t^\mu \mathbf{K}_t dB_t^\mu \right] + E^R \left[ \int_0^T (\mathbf{H}')_t^{tr} C_t^{\mu'} \mathbf{K}'_t dB_t^{\mu'} \right] + E^R \left[ \int_0^T (\mathbf{H}'')_t^{tr} C_t^{\mu''} \mathbf{K}''_t dB_t^{\mu''} \right].$$

We refer to (2) for the notations in the addends of the right-hand side, so that for example

$$B_t^\mu := \sum_{i=1}^r \langle \mu^i \rangle_t^{R, \mathbb{A}} \quad c_{ij}^\mu(t) := \frac{d \langle \mu^i, \mu^j \rangle_t^{R, \mathbb{A}}}{dB_t^\mu}, \quad i, j \in (1, \dots, r).$$

As a consequence

$$\|(\mathbf{H}, \mathbf{H}', \mathbf{H}'')\|_{\mathcal{L}^2(\boldsymbol{\mu}, R, \mathbb{A}) \times \mathcal{L}^2(\boldsymbol{\mu}', R, \mathbb{A}) \times \mathcal{L}^2(\boldsymbol{\mu}'', R, \mathbb{A})}^2 := \|\mathbf{H}\|_{\mathcal{L}^2(\boldsymbol{\mu}, R, \mathbb{A})}^2 + \|\mathbf{H}'\|_{\mathcal{L}^2(\boldsymbol{\mu}', R, \mathbb{A})}^2 + \|\mathbf{H}''\|_{\mathcal{L}^2(\boldsymbol{\mu}'', R, \mathbb{A})}^2$$

with

$$\|\mathbf{H}\|_{\mathcal{L}^2(\boldsymbol{\mu}, R, \mathbb{A})}^2 = E^R \left[ \int_0^T \mathbf{H}_t^{tr} C_t^\mu \mathbf{H}_t dB_t^\mu \right]$$

that is

$$\|\mathbf{H}\|_{\mathcal{L}^2(\boldsymbol{\mu}, R, \mathbb{A})}^2 = E^R [\langle \mathbf{H} \bullet \boldsymbol{\mu} \rangle_T] = E^R [\langle \mathbf{H} \bullet \boldsymbol{\mu} \rangle_T^{R, \mathbb{A}}] = E^R [(\mathbf{H} \bullet \boldsymbol{\mu})_T^2].$$

The first equality derives from the general formula (4), the second and the last equalities derive from a characterization and from the definition of the  $(R, \mathbb{A})$ -sharp variation process of  $\mathbf{H} \bullet \boldsymbol{\mu}$  respectively.

Similarly

$$\|\mathbf{H}'\|_{\mathcal{L}^2(\boldsymbol{\mu}', R, \mathbb{A})}^2 = E^R [(\mathbf{H}' \bullet \boldsymbol{\mu}')_T^2], \quad \|\mathbf{H}''\|_{\mathcal{L}^2(\boldsymbol{\mu}'', R, \mathbb{A})}^2 = E^R [(\mathbf{H}'' \bullet \boldsymbol{\mu}'')_T^2].$$

Therefore it holds

$$\|(\mathbf{H}, \mathbf{H}', \mathbf{H}'')\|_{\mathcal{L}^2(\boldsymbol{\mu}, R, \mathbb{A}) \times \mathcal{L}^2(\boldsymbol{\mu}', R, \mathbb{A}) \times \mathcal{L}^2(\boldsymbol{\mu}'', R, \mathbb{A})}^2 = E^R [(\mathbf{H} \bullet \boldsymbol{\mu})_T^2 + (\mathbf{H}' \bullet \boldsymbol{\mu}')_T^2 + (\mathbf{H}'' \bullet \boldsymbol{\mu}'')_T^2].$$

At the same time (see (6))

$$\|\mathbf{H} \bullet \boldsymbol{\mu} + \mathbf{H}' \bullet \boldsymbol{\mu}' + \mathbf{H}'' \bullet \boldsymbol{\mu}''\|_{\mathcal{M}^2(R, \mathbb{A})}^2 = E^R \left[ \left( (\mathbf{H} \bullet \boldsymbol{\mu})_T + (\mathbf{H}' \bullet \boldsymbol{\mu}')_T + (\mathbf{H}'' \bullet \boldsymbol{\mu}'')_T \right)^2 \right].$$

By point i)

$$\mathbf{H} \bullet \boldsymbol{\mu} \cdot \mathbf{H}' \bullet \boldsymbol{\mu}', \quad \mathbf{H} \bullet \boldsymbol{\mu} \cdot \mathbf{H}'' \bullet \boldsymbol{\mu}'', \quad \mathbf{H}' \bullet \boldsymbol{\mu}' \cdot \mathbf{H}'' \bullet \boldsymbol{\mu}''$$

are centered  $(R, \mathbb{A})$ -martingales and in particular

$$E^R [(\mathbf{H} \bullet \boldsymbol{\mu})_T \cdot (\mathbf{H}' \bullet \boldsymbol{\mu}')_T] = E^R [(\mathbf{H} \bullet \boldsymbol{\mu})_T \cdot (\mathbf{H}'' \bullet \boldsymbol{\mu}'')_T] = E^R [(\mathbf{H}' \bullet \boldsymbol{\mu}')_T \cdot (\mathbf{H}'' \bullet \boldsymbol{\mu}'')_T] = 0.$$

Then

$$\|\mathbf{H} \bullet \boldsymbol{\mu} + \mathbf{H}' \bullet \boldsymbol{\mu}' + \mathbf{H}'' \bullet \boldsymbol{\mu}''\|_{\mathcal{M}^2(R, \mathbb{A})}^2 = E^R [(\mathbf{H} \bullet \boldsymbol{\mu})_T^2 + (\mathbf{H}' \bullet \boldsymbol{\mu}')_T^2 + (\mathbf{H}'' \bullet \boldsymbol{\mu}'')_T^2]$$

and therefore we get

$$\|(\mathbf{H}, \mathbf{H}', \mathbf{H}'')\|_{\mathcal{L}^2(\boldsymbol{\mu}, R, \mathbb{A}) \times \mathcal{L}^2(\boldsymbol{\mu}', R, \mathbb{A}) \times \mathcal{L}^2(\boldsymbol{\mu}'', R, \mathbb{A})}^2 = \|\mathbf{H} \bullet \boldsymbol{\mu} + \mathbf{H}' \bullet \boldsymbol{\mu}' + \mathbf{H}'' \bullet \boldsymbol{\mu}''\|_{\mathcal{M}^2(R, \mathbb{A})}^2.$$

Then  $\mathcal{I}$  is closed. For proving that  $\mathcal{I}$  is stable it is sufficient to recall that its elements are sum of elements of stable subspaces.

The uniqueness of the triplet follows observing that if it would exists a different triplet  $\boldsymbol{\eta}, \boldsymbol{\eta}', \boldsymbol{\eta}''$  such that

$$\Theta \bullet (\boldsymbol{\mu}, \boldsymbol{\mu}', \boldsymbol{\mu}'') = \boldsymbol{\eta} \bullet \boldsymbol{\mu} + \boldsymbol{\eta}' \bullet \boldsymbol{\mu}' + \boldsymbol{\eta}'' \bullet \boldsymbol{\mu}''$$

then

$$\|(\boldsymbol{\xi} \bullet \boldsymbol{\mu} + \boldsymbol{\xi}' \bullet \boldsymbol{\mu}' + \boldsymbol{\xi}'' \bullet \boldsymbol{\mu}'') - (\boldsymbol{\eta} \bullet \boldsymbol{\mu} + \boldsymbol{\eta}' \bullet \boldsymbol{\mu}' + \boldsymbol{\eta}'' \bullet \boldsymbol{\mu}'')\|_{\mathcal{M}^2(R, \mathbb{A})}^2 = 0.$$

The linearity of the vector integral and point i) would imply

$$E^R[(\boldsymbol{\xi} - \boldsymbol{\eta}) \bullet \boldsymbol{\mu}_T^2] + E^R[(\boldsymbol{\xi}' - \boldsymbol{\eta}') \bullet \boldsymbol{\mu}'_T^2] + E^R[(\boldsymbol{\xi}'' - \boldsymbol{\eta}'') \bullet \boldsymbol{\mu}''_T^2] = 0.$$

□

**Remark 3.2.** Taking into account that the stable space generated by a martingale coincides with the set of vector integrals with respect to the martingale, we can summarize point i) and point ii) of the above lemma as follows

$$\mathcal{Z}^2(\boldsymbol{\mu}') \subset \mathcal{Z}^2(\boldsymbol{\mu})^\perp$$

$$\mathcal{Z}^2(\boldsymbol{\mu}, \boldsymbol{\mu}', \boldsymbol{\mu}'') = \mathcal{Z}^2(\boldsymbol{\mu}) \oplus \mathcal{Z}^2(\boldsymbol{\mu}') \oplus \mathcal{Z}^2(\boldsymbol{\mu}'').$$

**Remark 3.3.** We remark that point ii) of Lemma 3.1 holds when considering a finite number  $d$  of multidimensional  $(R, \mathbb{A})$ -martingales  $\boldsymbol{\mu}^i$ ,  $i = 1, \dots, d$  with mutually pairwise strongly orthogonal components. More precisely the following equality holds

$$\mathcal{Z}^2(\boldsymbol{\mu}^1, \dots, \boldsymbol{\mu}^d) = \bigoplus_{i=1}^d \mathcal{Z}^2(\boldsymbol{\mu}^i).$$

**Theorem 3.4.** Assume **A1)**, **A2)** and **A3)**. Then

- i1)  $\mathcal{F}_T$  and  $\mathcal{H}_T$  are  $P$ -independent;
- i2)  $\mathbb{G}$  fulfills the standard hypotheses;
- i3) every  $W$  in  $\mathcal{M}^2(P, \mathbb{G})$  can be uniquely represented as

$$W_t = W_0 + (\boldsymbol{\gamma}^W \bullet \mathbf{M})_t + (\boldsymbol{\kappa}^W \bullet \mathbf{N})_t + (\boldsymbol{\phi}^W \bullet [\mathbf{M}, \mathbf{N}]^\mathbf{V})_t, \quad P\text{-a.s.} \quad (14)$$

with  $\boldsymbol{\gamma}^W$  in  $\mathcal{L}^2(\mathbf{M}, P, \mathbb{G})$ ,  $\boldsymbol{\kappa}^W$  in  $\mathcal{L}^2(\mathbf{N}, P, \mathbb{G})$  and  $\boldsymbol{\phi}^W$  in  $\mathcal{L}^2([\mathbf{M}, \mathbf{N}]^\mathbf{V}, P, \mathbb{G})$ ;

- i4) there exists a probability measure  $Q$  on  $(\Omega, \mathcal{G}_T)$  such that  $(\mathbf{X}, \mathbf{Y}, [\mathbf{X}, \mathbf{Y}]^\mathbf{V})$  enjoys the  $(Q, \mathbb{G})$ -p.r.p.. More precisely every  $Z$  in  $\mathcal{M}^2(Q, \mathbb{G})$  can be uniquely represented as

$$Z_t = Z_0 + (\boldsymbol{\eta}^Z \bullet \mathbf{X})_t + (\boldsymbol{\theta}^Z \bullet \mathbf{Y})_t + (\boldsymbol{\zeta}^Z \bullet [\mathbf{X}, \mathbf{Y}]^\mathbf{V})_t \quad Q\text{-a.s.},$$

with  $\boldsymbol{\eta}^Z$  in  $\mathcal{L}^2(\mathbf{X}, Q, \mathbb{G})$ ,  $\boldsymbol{\theta}^Z$  in  $\mathcal{L}^2(\mathbf{Y}, Q, \mathbb{G})$  and  $\boldsymbol{\zeta}^Z$  in  $\mathcal{L}^2([\mathbf{X}, \mathbf{Y}]^\mathbf{V}, Q, \mathbb{G})$ .

*Proof.* i1) This statement is the extension to the multidimensional case of Lemma 4.2 in [6]. Thanks to point i) of Lemma 3.1 its proof is exactly the same. For the sake of completeness we repeat it here.

Proposition 2.1 implies that if  $A \in \mathcal{F}_T$  and  $B \in \mathcal{H}_T$  then

$$\mathbb{I}_A = P(A) + (\xi^A \bullet \mathbf{M})_T, \quad \mathbb{I}_B = P(B) + (\xi^B \bullet \mathbf{N})_T, \quad P\text{-a.s.} \quad (15)$$

for  $\xi^A$  and  $\xi^B$  in  $\mathcal{L}^2(\mathbf{M}, P, \mathbb{F})$  and  $\mathcal{L}^2(\mathbf{N}, P, \mathbb{H})$  respectively. These equalities imply that  $P(A \cap B)$  differs from  $P(A)P(B)$  by the expression

$$P(B)E^P[(\xi^A \bullet \mathbf{M})_T] + P(A)E^P[(\xi^B \bullet \mathbf{N})_T] + E^P[(\xi^A \bullet \mathbf{M})_T \cdot (\xi^B \bullet \mathbf{N})_T].$$

The above expression is null. In fact the  $(P, \mathbb{G})$ -martingale property of  $\mathbf{M}$  and  $\mathbf{N}$  and the integrability of the integrands  $\xi^A$  and  $\xi^B$  imply that the processes  $\xi^A \bullet \mathbf{M}$  and  $\xi^B \bullet \mathbf{N}$  are real centered martingales. Moreover, thanks to the assumption **A3**), by suitably applying point i) of Lemma 3.1, we get that the product  $\xi^A \bullet \mathbf{M} \cdot \xi^B \bullet \mathbf{N}$  is a centered real  $(P, \mathbb{G})$ -martingale.

- i2) This is a direct consequence of the previous point and Lemma 2.2 in [2].
- i3) The proof of this point will be done in three steps:
- (a) the first goal is to prove the  $(P, \mathbb{G})$ -p.r.p. for  $(\mathbf{M}, \mathbf{N}, [\mathbf{M}, \mathbf{N}]^\mathbf{V})$ ;
  - (b) as a second step the following key result is proved: fixed  $i \in (1, \dots, m)$  and  $j \in (1, \dots, n)$  the martingale  $[M^i, N^j]$  is  $(P, \mathbb{G})$ -strongly orthogonal to  $M^l$  and to  $N^h$  for arbitrary  $l \in (1, \dots, m)$  and  $h \in (1, \dots, n)$ ;
  - (c) finally point (a) and point (b) together with the second part of Lemma 3.1 allow to derive the result.

(a) The  $(P, \mathbb{G})$ -p.r.p. for  $(\mathbf{M}, \mathbf{N}, [\mathbf{M}, \mathbf{N}]^\mathbf{V})$  is achieved by proving that

$$\mathbb{P}((\mathbf{M}, \mathbf{N}, [\mathbf{M}, \mathbf{N}]^\mathbf{V}), \mathbb{G}) = \{P\},$$

or, equivalently, that for any  $R \in \mathbb{P}((\mathbf{M}, \mathbf{N}, [\mathbf{M}, \mathbf{N}]^\mathbf{V}), \mathbb{G})$ ,  $P$  and  $R$  coincide on the  $\pi$ -system

$$\{A \cap B, A \in \mathcal{F}_T, B \in \mathcal{H}_T\},$$

which generates  $\mathcal{G}_T$ . To this end, note that the equalities in (15) hold under  $R$  so that  $R(A \cap B)$  differs from  $P(A)P(B)$  by the expression

$$P(B)E^R[(\xi^A \bullet \mathbf{M})_T] + P(A)E^R[(\xi^B \bullet \mathbf{N})_T] + E^R[(\xi^A \bullet \mathbf{M})_T \cdot (\xi^B \bullet \mathbf{N})_T]. \quad (16)$$

The above expression is null. In fact **A1**) implies  $R|_{\mathcal{F}_T} = P|_{\mathcal{F}_T}$  and  $R|_{\mathcal{H}_T} = P|_{\mathcal{H}_T}$  and together with i1) this in turn implies that  $\xi^A \bullet \mathbf{M}$  and  $\xi^B \bullet \mathbf{N}$  are centered  $(R, \mathbb{G})$ -martingales. Moreover, by definition of  $R$ , for all  $i \in (1, \dots, m)$  and  $j \in (1, \dots, n)$  the process  $[M^i, N^j]$  is a  $(R, \mathbb{G})$ -martingale so that by point i) of Lemma 3.1 the product  $\xi^A \bullet \mathbf{M} \cdot \xi^B \bullet \mathbf{N}$  is a centered real  $(R, \mathbb{G})$ -martingale.

- (b)  $[M^i, N^j]$  is  $(P, \mathbb{G})$ -strongly orthogonal to the  $(P, \mathbb{G})$ -martingales  $M^l$  and  $N^h$ ,

if and only if  $[M^l, [M^i, N^j]]$  and  $[N^h, [M^i, N^j]]$  are uniformly integrable  $(P, \mathbb{G})$ -martingales.

Recall that

$$[M^i, N^j]_t = \langle M^{c,i}, N^{c,j} \rangle_t^{P, \mathbb{G}} + \sum_{s \leq t} \Delta M_s^i \Delta N_s^j, \quad P\text{-a.s.} \quad (17)$$

where  $M^{c,i}$  and  $N^{c,j}$  are the  $i$ -component of the continuous martingale part of  $\mathbf{M}$  and the  $j$ -component of the continuous martingale part of  $\mathbf{N}$  respectively. By point i1)  $M^{c,i}$  and  $N^{c,j}$  are independent  $(P, \mathbb{G})$ -martingales so that  $\langle M^{c,i}, N^{c,j} \rangle^{P, \mathbb{G}} \equiv 0$ , since by definition  $\langle M^{c,i}, N^{c,j} \rangle^{P, \mathbb{G}}$  is the unique  $\mathbb{G}$ -predictable process with finite variation such that  $M^{c,i} N^{c,j} - \langle M^{c,i}, N^{c,j} \rangle^{P, \mathbb{G}}$  is  $(P, \mathbb{G})$ -local martingale equal to 0 at time 0 (see Subsection 9.3.2. in [24]). Therefore

$$[M^i, N^j]_t = \sum_{s \leq t} \Delta M_s^i \Delta N_s^j. \quad (18)$$

As a consequence

$$[M^l, [M^i, N^j]]_t = \sum_{s \leq t} \Delta M_s^l \Delta M_s^i \Delta N_s^j.$$

Then for  $u \leq t$  one has

$$\begin{aligned} & E^P \left[ [M^l, [M^i, N^j]]_t \mid \mathcal{G}_u \right] \\ &= E^P \left[ \sum_{s \leq u} \Delta M_s^l \Delta M_s^i \Delta N_s^j \mid \mathcal{G}_u \right] + E^P \left[ \sum_{u < s \leq t} \Delta M_s^l \Delta M_s^i \Delta N_s^j \mid \mathcal{G}_u \right] \\ &= [M^l, [M^i, N^j]]_u + \sum_{u < s \leq t} E^P \left[ \Delta M_s^l \Delta M_s^i \Delta N_s^j \mid \mathcal{G}_u \right] \\ &= [M^l, [M^i, N^j]]_u + \sum_{u < s \leq t} E^P \left[ \Delta M_s^l \Delta M_s^i \mid \mathcal{F}_u \right] E^P \left[ \Delta N_s^j \mid \mathcal{H}_u \right], \end{aligned}$$

where the last equality follows by point i1) and Lemma 4.3 in [6]. The  $(P, \mathbb{G})$ -martingale property for  $[M^l, [M^i, N^j]]$  follows by observing that  $E^P [\Delta N_s^j \mid \mathcal{H}_u] = 0$ , for any  $s > u$ . Finally  $[M^l, [M^i, N^j]]$  is uniformly integrable, since it is a  $(P, \mathbb{G})$ -regular martingale.

Analogously one gets that  $[M^i, N^j]$  is  $(P, \mathbb{G})$ -strongly orthogonal to  $N^h$ .

(c) Point (a) implies that for every  $W \in \mathcal{M}^2(P, \mathbb{G})$  there exists a process  $\Theta^W$  in  $\mathcal{L}^2((\mathbf{M}, \mathbf{N}, [\mathbf{M}, \mathbf{N}]^{\mathbf{V}}), P, \mathbb{G})$  such that

$$W_t = W_0 + (\Theta^W \bullet (\mathbf{M}, \mathbf{N}, [\mathbf{M}, \mathbf{N}]^{\mathbf{V}}))_t.$$

Taking into account point b), we then apply point ii) of Lemma 3.1 with  $\mu = \mathbf{M}$ ,  $\mu' = \mathbf{N}$ ,  $\mu'' = [\mathbf{M}, \mathbf{N}]^{\mathbf{V}}$  and we obtain immediately the thesis.

i4) Define  $Q$  on  $(\Omega, \mathcal{G}_T)$  by

$$\frac{dQ}{dP} := L^{\mathbf{X}} \cdot L^{\mathbf{Y}}$$

where

$$L^{\mathbf{X}} := \frac{dP^{\mathbf{X}}}{dP|_{\mathcal{F}_T}}, \quad L^{\mathbf{Y}} := \frac{dP^{\mathbf{Y}}}{dP|_{\mathcal{H}_T}}.$$

The definition is well-posed since by point i1)  $L^{\mathbf{X}} \cdot L^{\mathbf{Y}}$  is in  $L^1(\Omega, P, \mathcal{G}_T)$ .  $L^{\mathbf{X}}$  and  $L^{\mathbf{Y}}$  are strictly positive and therefore  $Q$  and  $P|_{\mathcal{G}_T}$  are equivalent measures. Moreover for all  $A$  in  $\mathcal{F}_T$  and  $B$  in  $\mathcal{H}_T$  it holds

$$Q(A \cap B) = E^P[\mathbb{I}_A L^{\mathbf{X}}] E^P[\mathbb{I}_B L^{\mathbf{Y}}],$$

since  $\mathcal{F}_T$  and  $\mathcal{H}_T$  are independent under  $P$ . Using the equalities  $E^P[L^{\mathbf{X}}] = 1 = E^P[L^{\mathbf{Y}}]$  one immediately gets the  $Q$ -independence of  $\mathcal{F}_T$  and  $\mathcal{H}_T$ .

Finally  $\mathbf{X}$  is a  $(Q, \mathbb{F})$ -martingale since  $Q|_{\mathbb{F}} = P^{\mathbf{X}}$  and it is also a  $(Q, \mathbb{G})$ -martingale by the  $Q$ -independence of  $\mathbb{F}$  and  $\mathbb{H}$ . Analogously it can be shown that  $\mathbf{Y}$  is a  $(Q, \mathbb{G})$ -martingale. Moreover the  $Q$ -independence of  $\mathcal{F}_T$  and  $\mathcal{H}_T$  implies the analogous of point (b) for  $\mathbf{X}$  and  $\mathbf{Y}$ , that is the  $(Q, \mathbb{G})$ -strong orthogonality of  $X^i$  and  $Y^j$ , for all  $i \in (1, \dots, m)$  and  $j \in (1, \dots, n)$ . The representation i4) then follows by using the same procedure as in point i3). □

## 4 Two applications

In this section we discuss two applications of Theorem 3.4.

The first application is the extension of the representation property i3) of Theorem 3.4 to the case

$$\mathbb{G} := \mathbb{F}^1 \vee \dots \vee \mathbb{F}^d$$

where, for all  $i = 1, \dots, d$ ,  $\mathbb{F}^i \subset \mathcal{F}$  is the reference filtration on  $(\Omega, \mathcal{F}, P)$  of a real square integrable martingale  $M^i$  enjoying the  $(P, \mathbb{F}^i)$ -p.r.p.

The second application proposes a martingale representation result closed to that given in the second part of Proposition 5.3 in [5]. The statement dealt with the representation under the historical measure  $P$  of every square-integrable martingale of a market with default time  $\tau$  when the available information was completed by the observation of the default occurrence. Here we work under very similar hypotheses, but we also assume the *immersion property* of the filtration of the market  $\mathbb{F}$  into the filtration  $\mathbb{G} = \bigcap_{s > \cdot} \mathcal{F}_s \vee \sigma(\tau \wedge s)$ , that is every  $(P, \mathbb{F})$ -square-integrable martingale is a  $(P, \mathbb{G})$ -square-integrable martingale too.

### 4.1 The case of the union of a finite number of filtrations

First of all we prove a martingale representation result for a reference filtration which is the union of just three filtrations. Then we extend it to the case of a reference filtration which is the union of any finite number of filtrations.

**Theorem 4.1.** *Let  $\mathbb{F}^1, \mathbb{F}^2, \mathbb{F}^3$  be three filtrations on the space  $(\Omega, \mathcal{F}, P)$ . For  $i = 1, 2, 3$ , let  $M^i$ , be a real square integrable  $(P, \mathbb{F}^i)$ -martingale. Assume that*

**B1)**  $\mathbb{P}(M^i, \mathbb{F}^i) = \{P|_{\mathcal{F}_T^i}\}, \quad i = 1, 2, 3 ;$



**B2)** for all pair  $(i,j)$  with  $i,j \in \{1,2,3\}$ ,  $M^i$  and  $M^j$  are  $(P, \mathbb{G})$ -strongly orthogonal martingales where

$$\mathbb{G} := \mathbb{F}^1 \vee \mathbb{F}^2 \vee \mathbb{F}^3 ;$$

**B3)**  $M^3$  is  $(P, \mathbb{G})$ -strongly orthogonal to the  $(P, \mathbb{G})$ -martingale  $[M^1, M^2]$ .

Then

j1)  $\mathcal{F}_T^1, \mathcal{F}_T^2, \mathcal{F}_T^3$  are  $P$ -independent  $\sigma$ -algebras;

j2)  $\mathbb{G}$  fulfills the standard hypotheses;

j3) every  $W$  in  $\mathcal{M}^2(P, \mathbb{G})$  can be uniquely represented  $P$ -a.s. as

$$W_t = W_0 + \sum_{i=1}^3 \int_0^t \Upsilon_s^{W,i} dM_s^i + \sum_{i,j \in (1,2,3), i < j} \int_0^t \Phi_s^{W,i,j} d[M^i, M^j]_s + \int_0^t \Psi_s^W d[[M^1, M^2], M^3]_s$$

with  $\Upsilon^{W,i}$  in  $\mathcal{L}^2(M^i, P, \mathbb{G})$ ,  $\Phi^{W,i,j}$  in  $\mathcal{L}^2([M^i, M^j], P, \mathbb{G})$  and  $\Psi^W$  in  $\mathcal{L}^2([[M^1, M^2], M^3], P, \mathbb{G})$ . In particular the family

$$(M^1, M^2, M^3, [M^1, M^2], [M^1, M^3], [M^2, M^3], [[M^1, M^2], M^3])$$

is a  $(P, \mathbb{G})$ -basis of real strongly orthogonal martingales.

*Proof.* j1) In order to show that  $\mathcal{F}_T^1, \mathcal{F}_T^2, \mathcal{F}_T^3$  are  $P$ -independent we observe that for any choice of  $A^1 \in \mathcal{F}_T^1$ ,  $A^2 \in \mathcal{F}_T^2$  and  $A^3 \in \mathcal{F}_T^3$  the value  $P(A^1 \cap A^2 \cap A^3)$  differs from  $P(A^1)P(A^2)P(A^3)$  by the expectation under  $P$  of the expression

$$\begin{aligned} & \sum_i k_i \int_0^T \xi_s^i dM_s^i + \sum_{i,j \in (1,2,3), i < j} k_{ij} \int_0^T \xi_s^i dM_s^i \cdot \int_0^T \xi_s^j dM_s^j \\ & + \int_0^T \xi_s^1 dM_s^1 \cdot \int_0^T \xi_s^2 dM_s^2 \cdot \int_0^T \xi_s^3 dM_s^3 \end{aligned} \quad (19)$$

where, for  $i = 1, 2, 3$ ,  $\xi^i \in \mathcal{L}^2(M^i, P, \mathbb{F}^i)$  is the process, whose existence follows by assumption **B1)**, such that

$$\mathbb{I}_{A^i} = P(A^i) + \int_0^T \xi_s^i dM_s^i, \quad P\text{-a.s.} \quad (20)$$

and  $k_i$ , for  $i = 1, 2, 3$ , and  $k_{ij}$ , for  $i, j \in (1, 2, 3), i < j$ , are suitable constants.

But actually the expectation under  $P$  of (19) is null.

In fact, first of all, for all  $i = 1, 2, 3$ , the process  $\int_0^t \xi_s^i dM_s^i$  is an element of  $\mathcal{Z}^2(M^i)$  and therefore it is a centered  $(P, \mathbb{F}^i)$ -martingale, so that

$$E^P \left[ \sum_i \int_0^T \xi_s^i dM_s^i \right] = 0.$$

Moreover assumption **B2)** joint with the first part of Lemma 3.1 provides the product of  $\int_0^\cdot \xi_s^i dM_s^i \cdot \int_0^\cdot \xi_s^j dM_s^j$  to be a centered real  $(P, \mathbb{F}^i \vee \mathbb{F}^j)$ -martingale for all  $i, j \in (1, 2, 3)$  with  $i \neq j$ , so that

$$E^P \left[ \sum_{i,j \in (1,2,3), i < j} \int_0^T \xi_s^i dM_s^i \cdot \int_0^T \xi_s^j dM_s^j \right] = 0.$$

Finally, by Theorem 3.4 with  $m = n = 1$  and  $X = M^1, Y = M^2$ , we derive that  $(P, \mathbb{F}^1 \vee \mathbb{F}^2)$ -martingale  $\int_0^\cdot \xi_s^1 dM_s^1 \cdot \int_0^\cdot \xi_s^2 dM_s^2$  belongs to  $\mathcal{Z}^2(M^1, M^2, [M^1, M^2])$ . From assumptions **B2)** and **B3)** the martingale  $M^3$  is  $(P, \mathbb{G})$ -strongly orthogonal to  $M^1, M^2$  and  $[M^1, M^2]$  so that any element of  $\mathcal{Z}^2(M^3)$  is  $(P, \mathbb{G})$ -strongly orthogonal to all elements of  $\mathcal{Z}^2(M^1, M^2, [M^1, M^2])$  (see point a) of Theorem 4.7, page 116 in [18]). As a consequence  $\int_0^\cdot \xi_s^1 dM_s^1 \cdot \int_0^\cdot \xi_s^2 dM_s^2 \cdot \int_0^\cdot \xi_s^3 dM_s^3$  is a centered real  $(P, \mathbb{G})$ -martingale so that

$$E^P \left[ \int_0^T \xi_s^1 dM_s^1 \cdot \int_0^T \xi_s^2 dM_s^2 \cdot \int_0^T \xi_s^3 dM_s^3 \right] = 0.$$

- j2) This fact is a direct consequence of the previous point and Lemma 2.2 in [2].
- j3) From the assumptions **B2)** and **B3)** it follows immediately that the processes

$$M^1, M^2, M^3, [M^1, M^2], [M^1, M^3], [M^2, M^3], [[M^1, M^2], M^3]$$

form a family of  $(P, \mathbb{G})$ -martingales. It is easy to prove that these martingales are pairwise  $(P, \mathbb{G})$ -strongly orthogonal. In fact using j1) and as in point (b) of the proof of part i3) of Theorem 3.4 we get that, fixed  $i, j \in (1, 2, 3)$ ,  $[M^i, M^j]$  coincides  $P$ -a.s. with  $\sum_{s \leq t} \Delta M_s^i \Delta M_s^j$  and in particular  $[M^i, M^j]$  has no continuous martingale part. As a consequence we derive that  $[M^l, [M^i, M^j]]$ ,  $l = 1, 2, 3$ , coincides with  $\sum_{s \leq t} \Delta M_s^l \Delta M_s^i \Delta M_s^j$  and, again using j1), we get that is a  $(P, \mathbb{G})$ -martingale, that is  $M^l$  is  $(P, \mathbb{G})$ -strongly orthogonal to  $[M^i, M^j]$ . Analogously since  $[M^1, [M^2, M^3]]$  has no continuous martingale part, point j1) allows to prove that it is  $(P, \mathbb{G})$ -strongly orthogonal to  $M^k$ , for all  $k = 1, 2, 3$  and to  $[M^i, M^j]$ , for all  $i, j \in (1, 2, 3)$ .

From Theorem 36 in [27] it follows that

$$\begin{aligned} & \mathcal{Z}^2(M^1, M^2, M^3, [M^1, M^2], [M^1, M^3], [M^2, M^3], [[M^1, M^2], M^3]) = \\ & \oplus_{i=1}^3 \mathcal{Z}^2(M^i) \oplus_{i,j \in (1,2,3), i < j} \mathcal{Z}^2([M^i, M^j]) \oplus \mathcal{Z}^2([M^1, M^2], M^3) \end{aligned}$$

where the symbol of direct sum refers to the uniqueness of the representation of any martingale in  $\mathcal{M}^2(P, \mathbb{G})$  and to the orthogonality of the considered stable spaces, that is to the  $(P, \mathbb{G})$ -strong orthogonality of all pair of their elements.

We now show that the vector martingale

$$(M^1, M^2, M^3, [M^1, M^2], [M^1, M^3], [M^2, M^3], [[M^1, M^2], M^3])$$

enjoys the  $(P, \mathbb{G})$ -p.r.p..

This shall imply that

$$\mathcal{M}^2(P, \mathbb{G}) = \oplus_{i \in (1,2,3)} \mathcal{Z}^2(M^i) \oplus_{i,j \in (1,2,3), i < j} \mathcal{Z}^2([M^i, M^j]) \oplus \mathcal{Z}^2([[M^1, M^2], M^3])$$

that is the thesis.

Indeed we prove that

$$\mathbb{P}(M^1, M^2, M^3, [M^1, M^2], [M^1, M^3], [M^2, M^3], [[M^1, M^2], M^3], \mathbb{G}) = \{P\}.$$

In fact, if  $R \in \mathbb{P}(M^1, M^2, M^3, [M^1, M^2], [M^1, M^3], [M^2, M^3], [[M^1, M^2], M^3], \mathbb{G})$ . Then  $P$  and  $R$  coincide on the  $\pi$ -system

$$\{A^1 \cap A^2 \cap A^3, A^i \in \mathcal{F}_T^i, i = 1, 2, 3\},$$

which generates  $\mathcal{G}_T$ . To this end, note that the equalities in (20) hold under  $R$ -a.s. so that  $R(A^1 \cap A^2 \cap A^3)$  differs from  $P(A^1)P(A^2)P(A^3)$  by the expectation under  $R$  of the expression (19). The last turns out to be null by the same arguments used in the proof of point j1) since assumption **B1**) implies  $R|_{\mathcal{F}_T^i} = P|_{\mathcal{F}_T^i}$ ,  $i = 1, 2, 3$ .  $\square$

We now extend the theorem to the case of a general finite number  $n > 3$  of martingales.

**Theorem 4.2.** *Let  $\mathbb{F}^1, \dots, \mathbb{F}^d$  be filtrations on the space  $(\Omega, \mathcal{F}, P)$ . Let  $M^i$  be a real square integrable  $(P, \mathbb{F}^i)$ -martingale, for  $i = 1, \dots, d$ . Assume that*

**C1)**  $\mathbb{P}(M^i, \mathbb{F}^i) = \{P|_{\mathcal{F}_T^i}\}$ ,  $i = 1, \dots, d$ ;

**C2)** for all  $k \in (2, \dots, d)$ , for all set of index  $(i_1, i_2, \dots, i_k)$  with  $i_1 < i_2 < \dots < i_k$ ,

$$[[[M^{i_1}, M^{i_2}], M^{i_3}], \dots, M^{i_k}]$$

is a  $(P, \mathbb{G})$ -martingale where

$$\mathbb{G} := \mathbb{F}^1 \vee \mathbb{F}^2 \dots \vee \mathbb{F}^d.$$

Then the family obtained as the union of these sets of martingales

$M_i, i \in (1, 2 \dots d)$

$[M^i, M^j], i, j \in (1, \dots, d), i < j,$

$[[M^i, M^j], M^k], i, j, k \in (1, \dots, d), i < j < k,$

$[[[M^i, M^j], M^k], M^l], i, j, k, l \in (1, \dots, d), i < j < k < l,$

$\dots,$

$[[[[M^1, M^2], M^3], M^4] \dots, M^d]$

is a  $(P, \mathbb{G})$ -basis of real strongly orthogonal martingales or equivalently

$$\begin{aligned} \mathcal{M}^2(P, \mathbb{G}) = & \oplus_{i \in (1, 2 \dots d)} \mathcal{Z}^2(M^i) \\ & \oplus_{i, j \in (1, 2 \dots d), i < j} \mathcal{Z}^2([M^i, M^j]) \\ & \oplus_{i, j, k \in (1, 2 \dots d), i < j < k} \mathcal{Z}^2([[M^i, M^j], M^k]) \\ & \oplus_{i, j, k, l \in (1, 2 \dots d), i < j < k < l} \mathcal{Z}^2([[[M^i, M^j], M^k], M^l]) \\ & \dots \\ & \oplus \mathcal{Z}^2([[[[M^1, M^2], M^3], \dots, M^d])). \end{aligned}$$

*Proof.* For  $d \leq 3$  the result follows by previous theorem. For  $d > 3$  the key point is the  $P$ -independence of the  $\sigma$ -algebras  $\mathcal{F}_T^1, \mathcal{F}_T^2, \dots, \mathcal{F}_T^d$  that is, for any choice of  $A^1 \in \mathcal{F}_T^1$ ,  $A^2 \in \mathcal{F}_T^2$  and  $A^d \in \mathcal{F}_T^d$ , the factorization of  $P(A^1 \cap A^2 \cap \dots \cap A^d)$ . In order to prove it we proceed by induction. The basis of the induction is the  $P$ -independence of  $\mathcal{F}_T^1$  and  $\mathcal{F}_T^2$ , which derives immediately from j1) of previous theorem since the assumption **C1)** implies assumption **B1)** and assumption **C2)** implies assumption **B2)**. Fixed  $m$  in  $(4, \dots, d-1)$ , the inductive hypothesis is the  $P$ -independence of  $\mathcal{F}_T^1, \mathcal{F}_T^2, \dots, \mathcal{F}_T^m$ .

Following an analogous proof to that of point j3) of Theorem 4.1 it can be easily showed

$$\begin{aligned} \mathcal{M}^2 \left( P, \bigvee_{i=1}^m \mathbb{F}^i \right) &= \oplus_{i \in (1, 2, \dots, m)} \mathcal{Z}^2(M^i) \\ &\quad \oplus_{i, j \in (1, 2, \dots, m), i < j} \mathcal{Z}^2([M^i, M^j]) \\ &\quad \oplus_{i, j, k \in (1, 2, \dots, m), i < j < k} \mathcal{Z}^2([[M^i, M^j], M^k]) \\ &\quad \oplus_{i, j, k, l \in (1, 2, \dots, m), i < j < k < l} \mathcal{Z}^2([[[M^i, M^j], M^k], M^l]) \\ &\quad \dots \\ &\quad \oplus \mathcal{Z}^2([[[[M^1, M^2], M^3], \dots, M^m]]). \end{aligned} \quad (21)$$

Now we prove the  $P$ -independence of  $\mathcal{F}_T^1, \mathcal{F}_T^2, \dots, \mathcal{F}_T^{m+1}$ . Fixed  $A^1 \in \mathcal{F}_T^1$ ,  $A^2 \in \mathcal{F}_T^2$  and  $A^{m+1} \in \mathcal{F}_T^{m+1}$ , obviously  $P(A^1 \cap A^2 \cap \dots \cap A^{m+1})$  differs from  $P(A^1) \cdot P(A^2) \cdot \dots \cdot P(A^{m+1})$  by the  $P$ -expectation of an expression containing, up to multiplicative constants, terms of the form

$$\int_0^T \xi_s^{i_1} dM_s^{i_1} \cdot \int_0^T \xi_s^{i_2} dM_s^{i_2} \cdot \int_0^T \xi_s^{i_p} dM_s^{i_p} \quad (22)$$

with  $p \leq m$  and  $i_1 < i_2 < \dots < i_p \in (1, 2, \dots, m)$ , where  $\xi^{i_r} \in \mathcal{L}^2(M^{i_r}, P, \mathbb{F}^{i_r})$ ,  $1 \leq r \leq p$ , and one term of the form

$$\int_0^T \xi_s^1 dM_s^1 \cdot \int_0^T \xi_s^2 dM_s^2 \cdot \int_0^T \xi_s^m dM_s^m \cdot \int_0^T \xi_s^{m+1} dM_s^{m+1} \quad (23)$$

where  $\xi^r \in \mathcal{L}^2(M^r, P, \mathbb{F}^r)$ ,  $1 \leq r \leq m+1$ .

By the inductive hypothesis the terms of the form (22) are the final values of centered  $(P, \mathbb{G})$ -martingales so that they have null  $P$ -expectation.

Moreover by (21) the product of the first  $m$  integrals in (23), which belongs to  $\mathcal{M}^2(P, \bigvee_{i=1}^m \mathbb{F}^i)$ , is a sum of integrals with respect to martingales of the family

$$\begin{aligned} &M_i, i \in (1, 2, \dots, m), \\ &[M^i, M^j], i, j \in (1, \dots, m), i < j, \\ &[[M^i, M^j], M^k], i, j, k \in (1, \dots, m), i < j < k, \\ &[[[M^i, M^j], M^k], M^l], i, j, k, l \in (1, \dots, m), i < j < k < l, \\ &\dots, \\ &[[[[M^1, M^2], M^3], M^4], \dots, M^m]. \end{aligned}$$

Then, since by **C2)** the stable space generated by the above family is orthogonal to the stable space generated by  $M^{m+1}$ , (23) is the final value of a centered  $(P, \mathbb{G})$ -martingale so that its  $P$ -expectation is equal to zero.  $\square$

**Remark 4.3.** We get that  $2^d - 1$  is the biggest possible value of the multiplicity of  $\mathbb{G}$  in the sense of Davis Varaiya (see [11]), that is  $2^d$  is the maximum spanning number of the economy given by the assets  $M^1, \dots, M^d$  (see [14]).

## 4.2 An example of credit risk modeling

The second part of Proposition 5.3 in [5] dealt with the martingale representation under the market measure  $P$  when the reference filtration  $\mathbb{F}$  was progressively enlarged by the occurrence of a default time  $\tau$ . More precisely a  $(P, \mathbb{G})$ -basis of real strongly orthogonal martingales when  $\mathcal{G}_t = \cap_{s>t} \mathcal{F}_s \vee \sigma(\tau \wedge s)$  was derived. The main hypotheses were the existence of a real  $(P, \mathbb{F})$ -martingale which was a basis (not necessarily continuous) for the  $(P, \mathbb{F})$ -local martingales and the equivalence, for all  $t$ , between the  $\mathcal{F}_t$ -conditional law of  $\tau$  and a fixed deterministic probability measure on  $\mathbb{R}^+$  without atoms,  $\nu$ . The key tool was the decoupling martingale preserving measure introduced by Grorud and Pontier in [16] and by Amendinger in [1].

In a previous paper (see [20]) Jeanblanc and Le Cam assumed the existence of a (possibly multidimensional) continuous semi-martingale  $S$  with the  $(P^*, \mathbb{F})$ -p.r.p., and moreover they assumed *Jacod hypothesis* (see [19]), that is just the absolute continuity with respect to  $\nu$  of the  $\mathcal{F}_t$ -conditional law of  $\tau$  for all  $t$ . For the sake of completeness we recall that a time  $\tau$  under Jacod hypothesis is called an *initial time*, independently of the nature of  $\nu$ . In [20]  $\nu$  was equal to Lebesgue measure and the authors identified a basis for the  $(P^*, \mathbb{G})$ -martingales (see Theorem 1 in [20] and the analogous Proposition 4.2 in [5]).

It is to stress that both papers worked under the so-called *density hypothesis*, that is  $\tau$  initial time and  $\nu$  with no atoms (see [15]). As it is well-known this assumption forces  $\tau$  to have no atoms and to *avoids*  $\mathbb{F}$ -stopping times (for any finite  $\mathbb{F}$ -stopping time  $T$  it holds  $\mathbb{P}(\tau = T) = 0$ , see Proposition 1 in [20]).

Both papers gave a kind of generalization of the martingale representation result obtained by Kusuoka when  $\mathbb{F}$  is equal to the natural filtration of a Brownian motion  $B$  (see [26]). To the density hypothesis actually Kusuoka added the immersion property, which was the necessary condition in order to get  $B$  as an element of a  $(P, \mathbb{G})$ -basis. The second element of the  $(P, \mathbb{G})$ -basis was the  $\mathbb{F}$ -conditional compensated default process. Without the immersion hypothesis the Brownian motion should have been substituted by the martingale part of its  $(P, \mathbb{G})$ -semi-martingale decomposition, analogously to what happened in [5] and [20].

We propose a similar result, when the asset is a general multidimensional semi-martingale enjoying the p.r.p. and  $\tau$  satisfies the same hypotheses as in Proposition 5.3 in [5]. Moreover we assume immersion property of  $\mathbb{F}$  in  $\mathbb{G}$  under  $P$ .

**Proposition 4.4.** *Given a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{F}, P)$ , let  $\tau$  be a continuous random time such that*

$$P(\tau \in \cdot \mid \mathcal{F}_t) \sim P(\tau \in \cdot), \text{ for every } t \in [0, T], \text{ } P\text{-a.s.} \quad (24)$$

*and  $\mathbf{X}$  be an  $m$ -dimensional  $(P, \mathbb{F})$ -semi-martingale like in (7) satisfying hypotheses **A1**) and **A2**). Consider the progressively enlarged filtration  $\mathbb{G}$  defined by*

$$\mathcal{G}_t := \cap_{s>t} \mathcal{F}_s \vee \sigma(\tau \wedge s).$$

Let  $\lambda$  be the real process

$$\lambda_t = \frac{p_t(t)}{P(\tau > t \mid \mathcal{F}_t)}, \quad t \in [0, T], \quad (25)$$

with the process  $(p_t(u))_u$  defined by

$$\int_t^T p_t(u) du = P(\tau > t \mid \mathcal{F}_t). \quad (26)$$

Assume  $\mathbb{F}$  to be  $P$ -immersed in  $\mathbb{G}$ . Then the pair

$$\left( \mathbf{M}, \mathbb{I}_{\{\tau \leq \cdot\}} - \int_0^{\tau \wedge \cdot} \lambda_u du \right)$$

is a  $(P, \mathbb{G})$ -basis of multidimensional martingales.

*Proof.* Let  $F$  be the continuous distribution function of  $\tau$  and let  $\mathbb{H}$  be the natural filtration of the process  $(\mathbb{I}_{\{\tau \leq t\}})_t$ . Let  $N$  be defined by

$$N_t = \mathbb{I}_{\{\tau \leq t\}} - \int_0^{\tau \wedge t} \frac{dF_u}{1 - F_u}.$$

$N$  is a real  $(P, \mathbb{H})$ -martingale enjoying the  $(P|_{\mathcal{H}_T}, \mathbb{H})$ -p.r.p. (see Proposition 7.2.2.1 and Proposition 7.2.5.1 in [24]).

Moreover  $\mathbf{M}$  enjoys the  $(P|_{\mathcal{F}_T}, \mathbb{F})$ -p.r.p. (see Proposition 2.1).

Now, the equivalence (24) implies the existence of a probability measure  $P^*$  on  $(\Omega, \mathcal{G}_T)$  under which  $\mathbb{F}$  and  $\sigma(\tau)$  are independent and such that  $P^*|_{\mathcal{F}_T} = P|_{\mathcal{F}_T}$  and  $P^*|_{\mathcal{H}_T} = P|_{\mathcal{H}_T}$  (see Proposition 3.1 in [1]). Then  $P^*$  decouples  $\mathbb{F}$  and  $\mathbb{H}$ , since  $\mathcal{H}_T = \sigma(\tau)$ , so that, for any  $i = 1, \dots, m$ ,  $M^i$  and  $N$  are  $(P^*, \mathbb{G})$ -strongly orthogonal martingales. Therefore Theorem 3.4 applies. Indeed  $N$  jumps at an  $\mathbb{H}$ -totally inaccessible time since the law of  $\tau$  has no atoms (see Remark 7.2.1.2 in [24] and IV 107 in [12])<sup>4</sup> and under  $P^*$  the filtrations  $\mathbb{F}$  and  $\mathbb{H}$  are independent so that  $[M^i, N] \equiv 0$ . We conclude that  $(\mathbf{M}, N)$  is a  $(P^*, \mathbb{G})$ -multidimensional basis.

Let us introduce

$$\tilde{L}^* = \frac{dP|_{\mathcal{G}_T}}{dP^*}.$$

Now we define  $\tilde{M}^i, i = 1, \dots, m$ , and  $\tilde{N}$  by

$$\begin{aligned} \tilde{M}_t^i &:= M_t^i - \int_0^t \frac{1}{\tilde{L}_{s-}^*} d\langle \tilde{L}^*, M^i \rangle_s^{P^*, \mathbb{G}}, \quad t \in [0, T], \\ \tilde{N}_t &:= N_t - \int_0^t \frac{1}{\tilde{L}_{s-}^*} d\langle \tilde{L}^*, N \rangle_s^{P^*, \mathbb{G}}, \quad t \in [0, T]. \end{aligned}$$

We observe that the processes  $\langle \tilde{L}^*, M^i \rangle^{P^*, \mathbb{G}}$  and  $\langle \tilde{L}^*, N \rangle^{P^*, \mathbb{G}}$  exist (see e.g. VII 39 in [13]). By Lemma 2.4 in [23] the pair  $(\mathbf{M}, \tilde{N})$  enjoys the  $(P, \mathbb{G})$ -p.r.p.

<sup>4</sup> we observe that the jump time of  $N$  is also a  $\mathbb{G}$ -totally inaccessible time thanks to Lemma 3.5 in [9], however this is not important here

In order to identify the pair  $(\tilde{\mathbf{M}}, \tilde{N})$ , we recall that, since  $\mathbb{F}$  is  $P$ -immersed in  $\mathbb{G}$ , the process

$$\mathbb{I}_{\{\tau \leq \cdot\}} - \int_0^{\tau \wedge \cdot} \lambda_u du$$

with  $\lambda$  defined by (25) and (26), is a  $(P, \mathbb{G})$ -martingale (see pag 429 of [24] or Proposition 6.3 in [22]).

Now the pair  $(\tilde{\mathbf{M}}, \tilde{N})$  coincides with  $(\mathbf{M}, \mathbb{I}_{\{\tau \leq \cdot\}} - \int_0^{\tau \wedge \cdot} \lambda_u du)$ .

In fact, for  $i = 1, \dots, m$

$$\tilde{M}_t^i - M_t^i = \int_0^t \frac{1}{\tilde{L}_{s-}^*} d\langle \tilde{L}^*, M^i \rangle_s^{P^*, \mathbb{G}}, \quad t \in [0, T]$$

and

$$\begin{aligned} \tilde{N}_t - \left( \mathbb{I}_{\{\tau \leq t\}} - \int_0^{\tau \wedge t} \lambda_u du \right) = \\ \int_0^{\tau \wedge t} \frac{dF_u}{1 - F_u} - \int_0^{\tau \wedge t} \lambda_u du + \int_0^t \frac{1}{\tilde{L}_{s-}^*} d\langle \tilde{L}^*, N \rangle_s^{P^*, \mathbb{G}}, \quad t \in [0, T] \end{aligned}$$

are predictable  $(P, \mathbb{G})$ -local martingales, so they have to be continuous (see Theorem 43 Chapter IV in [25]). Moreover they have finite variation, so that they are necessarily null.

It remains to prove that, for all  $i = 1, \dots, m$ ,  $M^i$  and  $\mathbb{I}_{\{\tau \leq \cdot\}} - \int_0^{\tau \wedge \cdot} \lambda_u du$  are  $(P, \mathbb{G})$ -strongly orthogonal martingales. We consider their quadratic-covariation process which coincides  $P$ -a.s. and  $P^*$ -a.s. with

$$[M^i, \mathbb{I}_{\{\tau \leq \cdot\}}]_t - \left[ M^i, \int_0^{\tau \wedge \cdot} \lambda_u du \right]_t, \quad t \in [0, T].$$

The first addend is null since  $\tau$  avoids  $\mathbb{F}$ -stopping times. The second addend is null since  $\tau$  totally inaccessible implies that  $\int_0^{\tau \wedge \cdot} \lambda_u du$  is continuous and of finite variation.  $\square$

## 5 Perspectives

The object of ongoing research is to deeper investigate the problem of martingale representation under the historical measure in markets driven by processes sharing accessible jumps times with positive probability. In particular the authors conjecture to extend the Kusuoka like representation result to the case of a default time  $\tau$  which doesn't satisfies the density hypothesis.

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